

FEEDBACK STABILIZATION OF A SIMPLIFIED MODEL OF FLUID-STRUCTURE INTERACTION ON A TREE

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ABSTRACT. In this paper we study the dynamic feedback stability for a simplified model of fluid-structure interaction on a tree. We prove that, under some conditions, the energy of the solutions of the system decay exponentially to zero when the time tends to infinity. Our technique is based on a frequency domain method and a special analysis for the resolvent.

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1. Introduction

First of all, we introduce some notations needed to formulate the problem under consideration, as introduced in [1] or in [12]. Let \mathcal{T} be a tree (i.e. a planar connected graph without closed paths). By the multiplicity of a vertex of \mathcal{T} we mean the number of edges that branch out from the vertex. If the multiplicity is equal to one, the vertex is called exterior; otherwise, it is said to be interior. We denote by $\{e_1, \dots, e_N\}$ the set of edges of \mathcal{T} and $\{a_1, \dots, a_{N+1}\}$ its set of vertices and we assume that a_1 is the root of \mathcal{T} which will be denoted by \mathcal{R} , that e_1 is the edge containing \mathcal{R} and a_2 is its vertex different from \mathcal{R} .

We denote by \mathcal{M} the set of the interior vertices of \mathcal{T} and by \mathcal{S} the set of the exterior vertices, except \mathcal{R} and denote $I_{\mathcal{M}}$ and $I_{\mathcal{S}}$ the sets of indices of interior and exterior vertices, except \mathcal{R} , respectively. Then $I = I_{\mathcal{M}} \cup I_{\mathcal{S}}$ is the set of indices of all vertices, except the root \mathcal{R} . We denote by J the set $\{1, \dots, N\}$ and for $k \in I$ we will denote by J_k the set of indices of edges adjacent to a_k . If $k \in \mathcal{M}$, then the indice of the unique element of J_k will be denoted by j_k .

Furthermore, the length of the edge e_j will be denoted by ℓ_j . Then e_j may be parametrized by its arc length by means of the function $\pi_j : [0, \ell_j] \rightarrow e_j$, $x_j \mapsto \pi_j(x_j)$, and sometimes identified with the interval $(0, \ell_j)$.

For a function $y : \mathcal{T} \rightarrow \mathbb{C}$ we set $y^j = y \circ \pi_j$ its restriction to the edge e_j and we will denote $y^j(x) = y^j(\pi_j(x))$ for any x in $(0, \ell_j)$.

The incidence matrix $D = (d_{kj})_{(N+1) \times N}$ is defined by

$$d_{kj} = \begin{cases} 1 & \text{if } \pi_j(\ell_j) = a_k, \\ -1 & \text{if } \pi_j(0) = a_k, \\ 0 & \text{otherwise.} \end{cases}$$

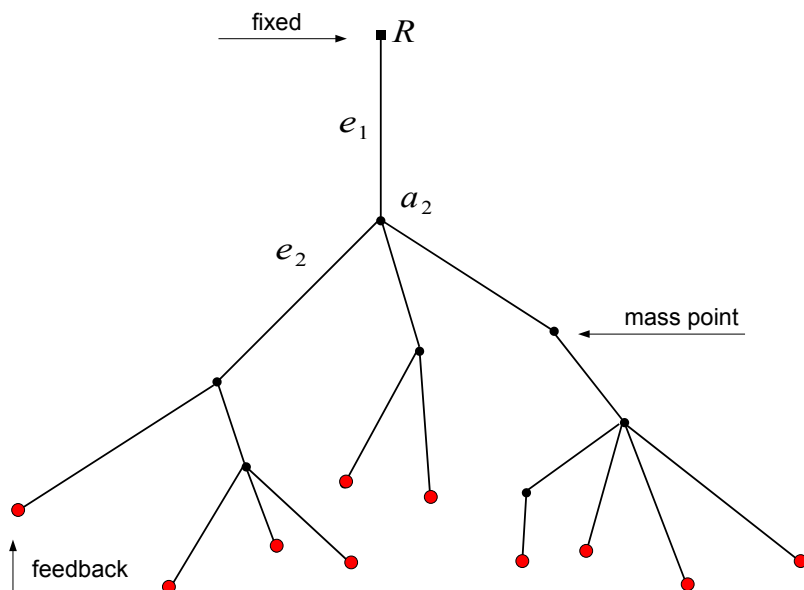
We denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the inner product and norm in L^2 -space, respectively.

In this paper, we study the stability of a model of fluid propagating in a 1- d network, under some feedback forces applied at exterior nodes, with the presence of point mass at inner nodes, see Figure 1. At rest, the network coincides with the tree \mathcal{T} .

More precisely, we consider the following initial and boundary value problem.

$$(1.1) \quad \begin{cases} y_{tt}^j - y_{xx}^j = 0 & \text{in } (0, \ell_j) \times (0, \infty), \quad j \in J, \\ \sum_{j \in J_k} d_{kj} y_x^j(a_k, t) = s'_k(t), & k \in I_{\mathcal{M}}, \\ s_k''(t) + s_k(t) = -y_t(a_k, t), & k \in I_{\mathcal{M}}, \\ y^j(a_k, t) = y^l(a_k, t), & j, l \in J_k, \quad k \in I_{\mathcal{M}}, \\ y^1(a_1, t) = 0, \\ d_{kj_k} y_x^{j_k}(a_k, t) = -y_t(a_k, t), & k \in I_{\mathcal{S}}, \\ s_k(0) = s_{k,0}, \quad s'_k(0) = s_{k,1}, & k \in I_{\mathcal{M}}, \\ y^j(x, 0) = y_0^j(x), \quad y_t^j(x, 0) = y_k^1(x), & x \in (0, \ell_j), \quad j \in J, \end{cases}$$

where $y^j : [0, \ell_j] \times (0, \infty) \rightarrow \mathbb{R}$, $j \in J$ represents the velocity potential of the fluid on the edge e_j and $s_k : (0, \infty) \rightarrow \mathbb{R}$, $k \in I_{\mathcal{M}}$ denotes the displacement of the point mass



occupying the node a_k . These functions allow us to identify the network with its rest graph. See [8] for more details.

This simplified model of fluid-structure interaction draws on the work of Ervedoza and Vanninathan [9]; they consider a fluid occupying a domain in two dimensions and a solid immersed in it and prove some results of controllability of such system; see also [16]. The problem of fluid structure interaction in one dimension has been studied by several authors. In [17] the authors study the asymptotic behavior of a one dimensional model of mass point moving in a fluid. They consider the same system in [18] but with a finite number of mass points floating in the fluid. Recently Tucsnak et al. [11] studied the controllability of a similar system.

Note that the point-wise (or boundary) stabilization on the wave equation has been treated during the last few years, see for example [2] for one string, and [3, 4, 5] for some networks of strings.

The main result of this paper asserts that, under some conditions, the energy of the solutions of the dissipative system decay exponentially to zero when the time tends to infinity. The method is based on a frequency domain method and a special analysis for the resolvent.

If $(y, s) = ((y^j)_{j \in J}, (s_k)_{k \in \mathcal{M}})$ is a solution of (1.1) we define the energy of (y, s) at instant t by

$$E(t) = \frac{1}{2} \sum_{j \in J} \int_0^{\ell_j} \left(\left| y_t^j(x, t) \right|^2 + \left| y_x^j(x, t) \right|^2 \right) dx + \frac{1}{2} \sum_{k \in I_M} \left(\left| s'_k(t) \right|^2 + \left| s_k(t) \right|^2 \right).$$

Simple formal calculations show that a sufficiently smooth solution of (1.1) satisfies the energy estimate

$$(1.2) \quad E(0) - E(t) = \sum_{k \in I_S} \int_0^t \left| y_t^{j_k}(a_k, s) \right|^2 ds, \quad \forall t \geq 0.$$

In particular (1.2) implies that

$$E(t) \leq E(0), \quad \forall t \geq 0.$$

Estimate above suggests that the natural wellposedness space for (1.1) is

$$H = V \times \prod_{j \in J} L^2(0, \ell_j) \times \left(\prod_{k \in I_{\mathcal{M}}} \mathbb{C} \right)^2$$

where

$$V = \left\{ \Phi \in \prod_{j \in J} H^1(0, \ell_j), \Phi^1(\mathcal{R}) = 0, \Phi^j(a_k) = \Phi^l(a_k), j, l \in J_k, k \in I_{\mathcal{M}} \right\}.$$

We can rewrite the system (1.1) as a first order differential equation, by putting

$$z(t) = \begin{pmatrix} y(t) \\ y'(t) \\ s(t) \\ s'(t) \end{pmatrix} :$$

$$z'(t) = \mathcal{A}z(t), \quad z(0) = z_0 = \begin{pmatrix} y_0 \\ y_1 \\ s^0 \\ s^1 \end{pmatrix},$$

where

$$\mathcal{A} \begin{pmatrix} y \\ v \\ p \\ q \end{pmatrix} = \begin{pmatrix} v \\ \frac{d^2 y}{dx^2} \\ q \\ -p - v_{\mathcal{M}} \end{pmatrix}, \forall \begin{pmatrix} y \\ v \\ p \\ q \end{pmatrix} \in \mathcal{D}(\mathcal{A}),$$

with $v_{\mathcal{M}} = (v(a_k))_{k \in I_{\mathcal{M}}}$, and

$$\mathcal{D}(\mathcal{A}) = \left\{ (y, v, p, q) \in \left[\prod_{j \in J} H^2(0, \ell_j) \cap V \right] \times V \times \left(\prod_{k \in I_{\mathcal{M}}} \mathbb{C} \right)^2 \text{ satisfying (1.3)} \right\}$$

where

$$(1.3) \quad \begin{cases} \sum_{j \in J_k} d_{kj} \frac{dy^j}{dx}(a_k) = q_k, \quad \forall k \in I_{\mathcal{M}}, \\ d_{kj} \frac{dy^{j_k}}{dx}(a_k) = -v^{j_k}(a_k), \quad \forall k \in I_S. \end{cases}$$

To simplify the notations, sometimes, we will write $y(a_k)$ instead of $y^j(a_k)$ for y in V .

The outline of this work is the following. In Section 2 we prove the existence and uniqueness of solutions for system (1.1). Section 3 is devoted to prove the exponential stability of the associated semi group. Finally, in Section 4, we prove the lack of exponential stability if the graph contain a circuit or if there is at least one uncontrolled

exterior node (other than the root). The section 5 is devoted to the study of the case of a chain with non equal mass points.

2. Wellposedness

Lemma 2.1. *The operator \mathcal{A} generates a \mathcal{C}_0 -semigroup of contractions $(S(t))_{t \in \mathbb{R}^+}$.*

Proof. It is clear that the operator \mathcal{A} is dissipative, moreover, for every $z = (y, v, p, q) \in \mathcal{D}(\mathcal{A})$,

$$(2.4) \quad \operatorname{Re}(\langle \mathcal{A}z, z \rangle_H) = - \sum_{k \in I_S} |v^{j_k}(a_k)|^2 \leq 0.$$

Now we prove that every positive real number λ belongs to $\rho(\mathcal{A})$, the resolvent set of \mathcal{A} . For this, let $Z = (f, g, c, d) \in H$ and we solve the equation

$$(2.5) \quad (\lambda - \mathcal{A})z = Z$$

with z in $\mathcal{D}(\mathcal{A})$.

We rewrite (2.5) explicitly

$$(2.6) \quad \begin{cases} \lambda y^j - v^j = f^j, & j \in J, \\ \lambda v^j - \frac{d^2 y^j}{dx^2} = g^j, & j \in J, \\ \lambda p_k - q_k = c_k, & k \in I_{\mathcal{M}}, \\ \lambda q_k + p_k + v_k(a_k) = d_k, & k \in I_{\mathcal{M}}. \end{cases}$$

We eliminate (v, q) in (2.6) to get

$$(2.7) \quad \lambda^2 y^j - \frac{d^2 y^j}{dx^2} = g^j + \lambda f^j, \quad j \in J,$$

$$(2.8) \quad (\lambda^2 + 1)p_k + v_k(a_k) = d_k + \lambda c_k, \quad k \in I_{\mathcal{M}}.$$

Let w in V . Multiplying (2.7) by w^j in $L^2(0, \ell_j)$ and summing over $j \in J$,

$$(2.9) \quad \begin{aligned} & \lambda^2 \sum_{j \in J} \left(\int_0^{\ell_j} y^j \overline{w^j} dx + \int_0^{\ell_j} \frac{dy^j}{dx} \frac{d\overline{w^j}}{dx} dx \right) - \sum_{k \in I_{\mathcal{M}}} \overline{w(a_k)} q_k + \sum_{k \in I_S} \overline{w(a_k)} (a_k) \\ &= \sum_{j \in J} \int_0^{\ell_j} (g^j + \lambda f^j) \overline{w^j} dx. \end{aligned}$$

Multiplying (2.8) by $r_k \in \mathbb{C}$ and summing over $k \in I_{\mathcal{M}}$, we get

$$(2.10) \quad (\lambda^2 + 1) \sum_{k \in I_{\mathcal{M}}} p_k \overline{r_k} + \sum_{k \in I_{\mathcal{M}}} v(a_k) \overline{r_k} = \sum_{k \in I_{\mathcal{M}}} (d_k + \lambda c_k) \overline{r_k}.$$

Summing (2.9) and (2.10) we get

$$\begin{aligned} & \lambda^2 \sum_{j \in J} \left(\langle y^j, w^j \rangle + \left\langle \frac{dy^j}{dx}, \frac{dw^j}{dx} \right\rangle \right) + (\lambda^2 + 1) \sum_{k \in I_{\mathcal{M}}} p_k \overline{r_k} + \sum_{k \in I_{\mathcal{M}}} \left(v(a_k) \overline{r_k} - \overline{w(a_k)} q_k \right) \\ &+ \sum_{k \in I_S} \overline{w(a_k)} v(a_k) = \sum_{j \in J} \langle g^j + \lambda f^j, w^j \rangle + \sum_{k \in I_{\mathcal{M}}} (d_k + \lambda c_k) \overline{r_k} \end{aligned}$$

to obtain

$$a((y, p), (w, r)) = f(w, r)$$

where

$$\begin{aligned} a((y, p), (w, r)) &= \lambda^2 \sum_{j \in J} \left(\langle y^j, w^j \rangle + \left\langle \frac{dy^j}{dx}, \frac{dw^j}{dx} \right\rangle \right) + (\lambda^2 + 1) \sum_{k \in I_{\mathcal{M}}} p_k \overline{r_k} \\ &\quad + \lambda \sum_{k \in I_{\mathcal{M}}} \left(y(a_k) \overline{r_k} - \overline{w(a_k)} p_k \right) + \lambda \sum_{k \in I_S} \overline{w(a_k)} y(a_k) \end{aligned}$$

and

$$\begin{aligned} f(w, r) &= \sum_{j \in J} \langle g^j + \lambda f^j, w^j \rangle + \sum_{k \in I_{\mathcal{M}}} (d_k + \lambda c_k) \overline{r_k} - \sum_{k \in I_{\mathcal{M}}} \overline{w(a_k)} c_k \\ &\quad + \sum_{k \in I_{\mathcal{M}}} f(a_k) \overline{r_k} + \sum_{k \in I_S} \overline{w(a_k)} f(a_k). \end{aligned}$$

a is a continuous sesquilinear form on $V \times \prod_{k \in I_{\mathcal{M}}} \mathbb{C}$ and f is a continuous anti-linear form on $V \times \prod_{k \in I_{\mathcal{M}}} \mathbb{C}$. Moreover

$$a((w, r), (w, r)) = \lambda^2 \sum_{j \in J} \left(\|w^j\|^2 + \left\| \frac{dw^j}{dx} \right\|^2 \right) + (\lambda^2 + 1) \sum_{k \in I_{\mathcal{M}}} |r_k|^2 - 2i \operatorname{Im} \left(\sum_{k \in I_{\mathcal{M}}} \overline{w(a_k)} r_k \right)$$

and then

$$|a((w, r), (w, r))| \geq \lambda^2 \left[\sum_{j \in J} \left(\|w^j\|^2 + \left\| \frac{dw^j}{dx} \right\|^2 \right) + \sum_{k \in I_{\mathcal{M}}} |r_k|^2 \right]$$

that is, a is coercive. The conclusion result immediately from the Lax Milgram lemma. \square

Proposition 2.2. *Suppose that $(y_0, y_1, s^0, s^1) \in H$. Then the problem (1.1) admits a unique solution*

$$(y, y', s, s') \in \mathcal{C}([0, +\infty); H).$$

If $(y_0, y_1, s^0, s^1) \in \mathcal{D}(\mathcal{A})$ then

$$(y, y', s, s') \in \mathcal{C}([0, +\infty), \mathcal{D}(\mathcal{A})) \cap \mathcal{C}^1([0, +\infty); \mathcal{H}).$$

Moreover (y, s) satisfies the energy estimate (1.2).

3. Exponential stability

It is clear that if \mathcal{T} contains an edge e_j , not attached to a leaf, with length $\ell_j \in \pi\mathbb{N}$ then \mathbf{i} is eigenvalue of \mathcal{A} with eigenvector $z = (y, v, p, q)$ such that $y^j = \mathbf{i} \sin x$, $v^j = -\sin x$, $p_k = 1$ and $q_k = \mathbf{i}$ when a_k is the nearest end of e_j to the root \mathcal{R} , and all the other components of z are null.

In the following, the tree \mathcal{T} is said to be a **Pi-tree** if it has no edges of length $m\pi$ with $m \in \mathbb{N}^*$, except maybe those attached to leaves. Then we have the following result:

Lemma 3.1. *The spectrum of \mathcal{A} contains no point on the imaginary axis if and only if \mathcal{T} is a Pi-tree.*

Proof. Since \mathcal{A} has compact resolvent, its spectrum $\sigma(\mathcal{A})$ only consists of eigenvalues of \mathcal{A} . We will show that the equation

$$(3.11) \quad \mathcal{A}z = \mathbf{i}\beta z$$

with $z = \begin{pmatrix} y \\ v \\ p \\ q \end{pmatrix} \in \mathcal{D}(\mathcal{A})$ and $\beta \in \mathbb{R}$ has only trivial solution.

By taking the inner product of (3.11) with $z \in H$ and using that

$$\operatorname{Re}(\langle \mathcal{A}z, z \rangle_H) = - \sum_{k \in I_S} |v(a_k)|^2$$

we obtain that $v(a_k) = 0$ for $k \in I_S$ and then,

$$(3.12) \quad \frac{dy}{dx}(a_k) = 0 \text{ for } k \in I_S.$$

Now the equation (3.11) can be rewritten explicitly as

$$(3.13) \quad v^j = \mathbf{i}\beta y^j, \quad j \in J,$$

$$(3.14) \quad \frac{d^2 y^j}{dx^2} = \mathbf{i}\beta v^j, \quad j \in J,$$

$$(3.15) \quad q_k = \mathbf{i}\beta p_k, \quad k \in I_{\mathcal{M}},$$

$$(3.16) \quad -p_k - v(a_k) = \mathbf{i}\beta q_k, \quad k \in I_{\mathcal{M}}.$$

If $\beta = 0$ then $v = 0$, $q = 0$ and $p = 0$.

Multiplying the second equation in the above system by y^j and then summing over j , we obtain

$$\sum_{j \in J} \left\| \frac{dy^j}{dx} \right\|^2 = 0,$$

which implies, using continuity condition of y at inner nodes and its Dirichlet condition at \mathcal{R} , that $y = 0$.

Next, we suppose that $\beta \neq 0$. We have, using (3.13-3.16),

$$(3.17) \quad \beta^2 y^j + \frac{d^2 y^j}{dx^2} = 0, \quad j \in J,$$

$$(3.18) \quad (\beta^2 - 1)p_k = v(a_k), \quad k \in I_{\mathcal{M}}.$$

The function y^j , $j \in J$ is then of the form

$$y^j = \alpha_1 \sin(\beta x) + \alpha_2 \cos(\beta x).$$

Using (3.12) and (3.13) we deduce that, for $k \in I_S$, $y^{j_k} = 0$ and then $v^{j_k} = 0$.

For the sequel of the proof we consider two cases:

First case: $\beta^2 = 1$. From (3.18), (3.13) and the Dirichlet condition at \mathcal{R} , we deduce that $y^j(0) = y^j(\ell_j) = 0$ for $j \in J$. Since \mathcal{T} is a Pi-tree we conclude that $y^j = 0$ for $j \in J$. Return back to the balance conditions and (3.15), one can deduce that $q = p = 0$ and hence $y = 0$.

Second case: $\beta^2 \neq 1$. Let a_k the second end of an edge e_j attached to a leaf. Then $p_k = 0$ and $q_k = 0$. Let e_l the edge ended by a_k and not attached to a leaf. We have $y^l(a_k) = 0$ and $\frac{dy^l}{dx}(a_k) = 0$. Then by (3.17) $y^l = 0$. We iterate such procedure from root to leaves to conclude that $y = 0$. \square

The main result of this paper concerns the precise asymptotic behavior of the solution of (1.1). Our technique is based on a frequency domain method and a special analysis for the resolvent.

Recall that the system (1.1) is said to be exponentially stable if there exist two constants $M, \omega > 0$, such that for all $(y_0, y_1, s^0, s^1) \in H$,

$$E(t) \leq M e^{-\omega t} \|(y_0, y_1, s^0, s^1)\|_H^2, \quad \forall t \geq 0.$$

Then, our main result is the following:

Theorem 3.2. *The system defined by equations (1.1) is exponentially stable if and only if \mathcal{T} is a Pi-tree.*

Proof. By classical result (see Huang [10] and Prüss [13]) it suffices to show that the corresponding operator \mathcal{A} satisfies the following two conditions:

$$(3.19) \quad \rho(\mathcal{A}) \supset \{\mathbf{i}\beta \mid \beta \in \mathbb{R}\} \equiv \mathbf{i}\mathbb{R},$$

and

$$(3.20) \quad \lim_{|\beta| \rightarrow \infty} \sup \|(\mathbf{i}\beta - \mathcal{A})^{-1}\| < \infty,$$

where $\rho(\mathcal{A})$ denotes the resolvent set of the operator \mathcal{A} .

By lemma 3.1 the condition (3.19) is satisfied. Suppose that the condition (3.20) is false. By the Banach-Steinhaus Theorem (see [7]), there exist a sequence of real numbers

$$\beta_n \rightarrow \infty \quad (\beta_n > 0 \text{ without loss of generality}) \text{ and a sequence of vector } z_n = \begin{pmatrix} y_n \\ v_n \\ p_n \\ q_n \end{pmatrix} \in$$

$\mathcal{D}(\mathcal{A})$ with $\|z_n\|_H = 1$ such that

$$(3.21) \quad \|(\mathbf{i}\beta_n I - \mathcal{A})z_n\|_H \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$

i.e.

$$(3.22) \quad \mathbf{i}\beta_n y_n^j - v_n^j \equiv f_n^j \longrightarrow 0 \quad \text{in } H^2(0, \ell_j),$$

$$(3.23) \quad \mathbf{i}\beta_n v_n^j - \frac{d^2 y_n^j}{dx^2} \equiv g_n^j \longrightarrow 0 \quad \text{in } L^2(0, \ell_j),$$

$$(3.24) \quad \mathbf{i}\beta_n p_{k,n} - q_{k,n} \equiv h_{k,n} \longrightarrow 0 \quad \text{in } \mathbb{C},$$

$$(3.25) \quad \mathbf{i}\beta_n q_{k,n} + p_{k,n} + v_n(a_k) \equiv r_{k,n} \longrightarrow 0 \quad \text{in } \mathbb{C}.$$

Our goal is to derive from (3.21) that $\|z_n\|_H$ converge to zero, thus, a contradiction.

The proof is divided in three steps:

First step. Recall that for every j in J ,

$$\frac{\|v^j\|_\infty}{\beta_n^{1/2}} \leq \frac{C_1}{\beta_n^{1/2}} \left\| \frac{dv^j}{dx} \right\|^{1/2} \|v^j\|^{1/2} + C_2 \frac{\|v^j\|}{\beta_n^{1/2}},$$

for some positives constants C_1 and C_2 . This implies, using (3.22), that $\frac{\|v^j\|_\infty}{\beta_n^{1/2}}$ is bounded.

Then, for every k in $I_{\mathcal{M}}$, by (3.25), $q_{k,n}$ converge to zero and then $\beta_n p_{k,n}$ converge to zero in view of (3.24). In particular $p_{k,n}$ converge to zero.

Second step. We notice that from (2.4) we have

$$(3.26) \quad \|(\mathbf{i}\beta_n I - \mathcal{A})z_n\|_H \geq |Re \langle \mathbf{i}\beta_n I - \mathcal{A} \rangle z_n, z_n \rangle_H| = \sum_{k \in I_S} |v_n(a_k)|^2.$$

Then, by (3.21)

$$|v_n(a_k)| \longrightarrow 0, \quad \forall k \in I_S.$$

This further leads to

$$(3.27) \quad |\beta_n y_n(a_k)| \longrightarrow 0, \quad \forall k \in I_S.$$

due to (3.22) and the trace theorem.

We have also

$$(3.28) \quad \frac{dy^{j_k}}{dx}(a_k) \longrightarrow 0, \quad \forall k \in I_S.$$

Third step. Substituting (3.22) into (3.23) and (3.24) into (3.25) to get

$$(3.29) \quad -\beta_n^2 y_n^j - \frac{d^2 y_n^j}{dx^2} = g_n^j + \mathbf{i}\beta_n f_n^j, \quad j \in J$$

$$(3.30) \quad -\beta_n^2 p_{k,n} + v_n(a_k) = r_{k,n} + \mathbf{i}\beta_n h_{k,n}.$$

Next, we take the inner product of (3.29) with $b \frac{dy_n^j}{dx}$ in $L^2(0, \ell_j)$ for $b \in \mathcal{C}^1([0, \ell_j])$ we get

$$\begin{aligned} \frac{1}{2} \beta_n^2 \left[|y_n^j(x)|^2 b(x) \right]_0^{\ell_j} + \frac{1}{2} \left[\left| \frac{dy_n^j}{dx}(x) \right|^2 b(x) \right]_0^{\ell_j} + \left[Re \left(\mathbf{i}\beta_n f_n^j(x) \overline{y_n^j}(x) b(x) \right) \right]_0^{\ell_j} \\ - \frac{1}{2} \int_0^{\ell_j} \left(|\beta_n y_n^j|^2 + \left| \frac{dy_n^j}{dx} \right|^2 \right) \frac{db}{dx} dx \longrightarrow 0. \end{aligned}$$

With (3.27) and (3.28), this leads to

$$\frac{1}{2} \int_0^{\ell_{j_k}} \left(|\beta_n y_n^{j_k}|^2 + \left| \frac{dy_n^{j_k}}{dx} \right|^2 \right) dx \longrightarrow 0$$

for every k in I_S , by taking $b = x$ or $b = \ell_{j_k} - x$. Moreover, as in [15] it follows that

$$\beta_n y_n(a_s) \longrightarrow 0, \quad \frac{dy_n^{j_k}}{dx}(a_s) \longrightarrow 0, \quad \text{and} \quad Re(i\beta_n f_n(a_s) \overline{y_n}(a_s)) \longrightarrow 0$$

where a_s is the end of e_{j_k} , different from a_k . We then conclude by iteration, that for every j in J ,

$$\int_0^{\ell_j} \left(|\beta_n y_n^j|^2 + |\partial_x y_n^j|^2 \right) dx \longrightarrow 0.$$

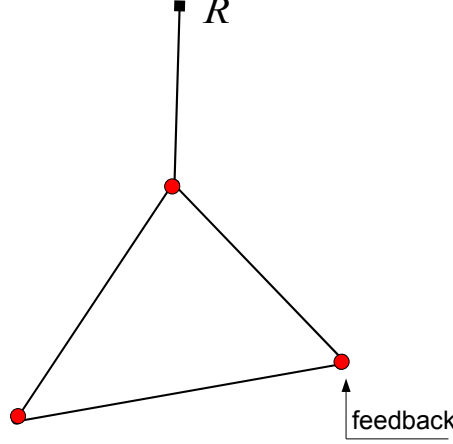


FIGURE 2. Circuit

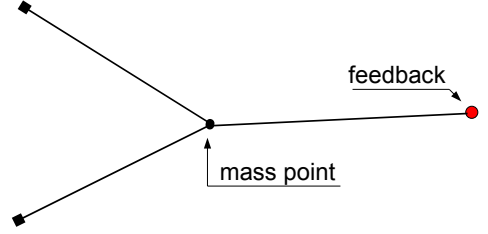


FIGURE 3. Start

Finally, in view of (3.22), we also get

$$\|v_n^j\| \longrightarrow 0, \text{ for } j \in J.$$

which implies that $\|z_n\|_H \longrightarrow 0$: clearly contradicts (3.21). \square

4. Some other cases

In this part we consider two particular cases. In the first, (Figure 2), there is a circuit in the graph. We prove that, even with much more controls, the exponential stability fails. The second case (Figure 3) proves that when we eliminate a control of a leaf in the initial case then the exponential stability fails.

4.1. A circuit. (Figure 2)

In this part we suppose that \mathcal{T} contains a circuit (Figure 2), with feedbacks at each inner node. Then the second equation in (1.1) will be

$$\sum_{j \in J_k} d_{kj} \partial_x y^j(a_k, t) = s'_1(t) - y_t(a_k, t), \quad a_k \in I_{\mathcal{M}}$$

and we can rewrite the system (1.1) in the Hilbert space H as

$$z'(t) = \mathcal{A}z(t).$$

Except that, here, the first equation in (1.3) will be replaced by the following

$$\sum_{j \in J_k} d_{kj} \frac{dy^j}{dx}(a_k) = q_k - y_t(a_k), \quad \forall k \in I_{\mathcal{M}}.$$

The operator \mathcal{A} generates a \mathcal{C}_0 -semigroup of contraction $(S(t))_{t \geq 0}$.

Next we suppose, without loss of generality, that $\ell_1 = \ell_2 = \ell_3 = 1$. We have the first result of asymptotic behavior

Theorem 4.1. *$(S(t))_{t \leq 0}$ is asymptotically stable if and only if ℓ_4 is irrational and not in $\pi\mathbb{Z}$.*

Proof. First, if ℓ_4 is in $\pi\mathbb{Z}$ then \mathbf{i} is an eigenvalue of \mathcal{A} (as in the case of a tree) and if $\ell_4 = \frac{a}{b}$ with a and b integer, then $\mathbf{i}b\pi$ is an eigenvalue of \mathcal{A} .

Now, we suppose that $\ell_4 \notin \mathbb{Q}$ and $\ell_4 \notin \pi\mathbb{Z}$. We only need to prove that $\mathbf{i}\mathbb{R} \subset \rho(\mathcal{A})$. For this, we will prove that the equation

$$(4.31) \quad \mathcal{A}z = \mathbf{i}\beta z$$

with $z = \begin{pmatrix} y \\ v \\ p \\ q \end{pmatrix} \in \mathcal{D}(\mathcal{A})$ and $\beta \in \mathbb{R}$ has only trivial solution.

we can show that $v(a_k) = 0$ for $k \in I_{\mathcal{M}}$. Now the equation (4.31) can be rewritten explicitly as

$$(4.32) \quad v^j = \mathbf{i}\beta y^j, \quad j \in J,$$

$$(4.33) \quad \begin{aligned} \frac{d^2 y^j}{dx^2} &= \mathbf{i}\beta v^j, \quad j \in J, \\ q_k &= \mathbf{i}\beta p_k, \quad k \in I_{\mathcal{M}}, \\ -p_k - v(a_k) &= \mathbf{i}\beta q_k, \quad k \in I_{\mathcal{M}}. \end{aligned}$$

If $\beta = 0$ then as in for the initial example we show that $y = 0$.

Next, we suppose that $\beta \neq 0$. We have $y(a_k) = 0$ for every $k \in I_{\mathcal{M}}$

$$(4.34) \quad \beta^2 y^j + \frac{d^2 y^j}{dx^2} = 0, \quad j \in J,$$

$$(4.35) \quad (\beta^2 - 1)p_k = 0, \quad k \in I_{\mathcal{M}}.$$

Then, since $y(a_k) = 0$ for every k then y^j is of the form

$$y^j = \alpha_j \sin(\beta x).$$

As in the case of a tree we consider two cases: $\beta^2 = 1$ and $\beta^2 \neq 1$

First case: $\beta^2 = 1$. we have $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and since $\ell_4 \notin \pi\mathbb{Z}$, $\alpha_4 = 0$. Return back to the balance conditions and (4.33), at inner nodes, one can deduce that $q = p = 0$ and hence $y = 0$.

Second case: $\beta^2 \neq 1$. we have $p_k = q_k = 0$ for every k in $I_{\mathcal{M}}$. Then $y = 0$. \square

Theorem 4.2. *The semigroup $(S(t))_{t \geq 0}$ is not exponentially stable, even if ℓ_4 is irrational and not in $\pi\mathbb{Z}$.*

Proof. To prove that $(S(t))_{t \geq 0}$ is not exponentially stable we consider the sequence f_n of vectors of H defined by $f_n = (0, g_n, 0, 0)$ where $g_n = (0, -\sin \beta_n x, 0, 0)$ and β_n is

a sequence of real numbers satisfying $\beta_n \mapsto +\infty$ and will be defined later. We then prove that the sequence $z_n = (y_n, v_n, p_n, q_n)$ of elements of $\mathcal{D}(\mathcal{A})$ such that

$$(\mathbf{i}\beta_n - \mathcal{A})z_n = f_n$$

is not bounded.

The sequence y_n should satisfy

$$\begin{cases} \beta_n^2 y_n^j + \partial_x^2 y_n^j = 0, & \text{for } j = 1, 3, 4 \\ \beta_n^2 y_n^2 + \partial_x^2 y_n^2 = \sin \beta_n x. \end{cases}$$

Then, for $j = 1, 3, 4$ there exists two complex numbers a_j and b_j such that

$$\begin{cases} y_n^j = a_j \sin(\beta_n x) + b_j \cos(\beta_n x), \\ \frac{dy_n^j}{dx} = -\beta_n b_j \sin(\beta_n x) + \beta_n a_j \cos(\beta_n x) \end{cases}$$

and there exists a_2 and b_2 in \mathbb{C} such that

$$\begin{cases} y_n^2 = a_2 \sin(\beta_n x) + (-\frac{x}{2\beta_n} + b_2) \cos(\beta_n x), \\ \frac{dy_n^2}{dx} = (\frac{x}{2} - \beta_n b_2) \sin(\beta_n x) + (-\frac{1}{2\beta_n} + \beta_n a_2) \cos(\beta_n x). \end{cases}$$

The boundary and transmission conditions are expressed as follows

$$\begin{cases} a_1 \sin(\beta_n) + b_1 \cos(\beta_n) = 0 \\ b_1 = b_2 = b_3 \\ -\frac{1}{2\beta_n} + \beta_n a_1 + \beta_n a_2 + \beta_n a_3 = -i\beta_n b_1 \\ a_2 \sin(\beta_n) + (-\frac{1}{2\beta_n} + b_2) \cos(\beta_n) = b_4 \\ \beta_n a_4 - \left((\frac{1}{2} - \beta_n b_2) \sin(\beta_n) + (-\frac{1}{2\beta_n} + \beta_n a_2) \cos(\beta_n) \right) = -i\beta_n b_4 \\ a_3 \sin(\beta_n) + b_1 \cos(\beta_n) = a_4 \sin(\beta_n \ell_4) + b_4 \cos(\beta_n \ell_4) \\ -\beta_n b_3 \sin(\beta_n) + \beta_n a_3 \cos(\beta_n) - \beta_n b_4 \sin(\beta_n \ell_4) + \beta_n a_4 \cos(\beta_n \ell_4) = \\ i\beta_n (a_3 \sin(\beta_n) + b_1 \cos(\beta_n)). \end{cases}$$

Then

$$\begin{cases} a_1 \sin(\beta_n) + b_1 \cos(\beta_n) = 0 \\ -\frac{1}{2\beta_n} + \beta_n a_1 + \beta_n a_2 + \beta_n a_3 = -i\beta_n b_1 \\ a_2 \sin(\beta_n) + (-\frac{1}{2\beta_n} + b_1) \cos(\beta_n) = b_4 \\ \beta_n a_4 - \left((\frac{1}{2} - \beta_n b_1) \sin(\beta_n) + (-\frac{1}{2\beta_n} + \beta_n a_2) \cos(\beta_n) \right) = -i\beta_n b_4 \\ a_3 \sin(\beta_n) + b_1 \cos(\beta_n) = a_4 \sin(\beta_n \ell_4) + b_4 \cos(\beta_n \ell_4) \\ \beta_n a_3 e^{-i\beta_n} - \beta_n b_4 \sin(\beta_n \ell_4) + \beta_n a_4 \cos(\beta_n \ell_4) = i\beta_n b_1 e^{-i\beta_n}. \end{cases}$$

Hence

$$(4.36) \quad (FB + AG)\beta_n b_1 = AH - F\beta_n C$$

where

$$\begin{aligned} A &= (1 + \cos(\beta_n \ell_4)) \sin(\beta_n) + e^{-i\beta_n} \sin(\beta_n \ell_4), \\ B &= (2 - \cos(\beta_n \ell_4) \cos(\beta_n) + \mathbf{i} \left(e^{-i\beta_n} \sin(\beta_n \ell_4) - \sin(\beta_n) \right)), \\ C &= \frac{1}{2\beta_n^2} \sin(\beta_n) - \left(\frac{1}{2} \sin(\beta_n) + \frac{1}{2} (-1 + \mathbf{i}) \cos(\beta_n) \right) \sin(\beta_n \ell_4) \\ &\quad + \frac{1}{2\beta_n} \cos(\beta_n) \cos(\beta_n \ell_4) \end{aligned}$$

and

$$\begin{aligned}
F &= e^{-i\beta_n} (\cos \beta_n - 1) - \sin(\beta_n) \sin(\beta_n \ell_4), \\
G &= e^{-i\beta_n} (\cot \beta_n - 2i) - \cos(\beta_n) \sin(\beta_n \ell_4) - i e^{-i\beta_n} \cos(\beta_n \ell_4), \\
H &= -\frac{1}{2\beta_n} e^{-i\beta_n} - \frac{1}{2\beta_n} \cos(\beta_n) \sin(\beta_n \ell_4) - \frac{1}{2} \sin(\beta_n) \cos(\beta_n \ell_4) \\
&\quad - \frac{1}{2} (-1 + i) \cos(\beta_n) \cos(\beta_n \ell_4).
\end{aligned}$$

Now by using the Asymptotic Dirichlet's theorem [14], there exists $(p_n, q_n) \in \mathbb{N}^2$ such that $\frac{p_n}{q_n}$ converge to ℓ_4 , p_n and q_n tend to infinity as n goes to infinity and for every n in \mathbb{N}

$$|q_n \ell_4 - p_n| < \frac{1}{q_n}.$$

Take $\beta_n = 2\pi q_n + \frac{2\pi}{q_n^{1/4}}$, then there exists a positive integer n_0 such that for every integer $n \geq n_0$,

$$0 < \lambda_n := -\frac{2\pi}{q_n} + \frac{2\pi \ell_4}{q_n^{1/4}} < \beta_n \ell_4 - 2\pi p_n < \mu_n := \frac{2\pi}{q_n} + \frac{2\pi \ell_4}{q_n^{1/4}} < \frac{\pi}{2}$$

and

$$\sin(\lambda_n) < \sin(\beta_n \ell_4) < \sin(\mu_n), \quad \cos(\lambda_n) < \cos(\beta_n \ell_4) < \cos(\mu_n).$$

Moreover $\sin(\beta_n \ell_4)$ and $\cos(\beta_n \ell_4)$ satisfy the following asymptotic approximations,

$$\sin(\beta_n \ell_4) = \frac{2\pi \ell_4}{q_n^{1/4}} + o\left(\frac{1}{q_n^{1/4}}\right), \quad \cos(\beta_n \ell_4) = 1 + o\left(\frac{1}{q_n^{1/4}}\right).$$

We have also

$$\sin(\beta_n) = \sin\left(\frac{2\pi}{q_n^{1/4}}\right) = \frac{2\pi}{q_n^{1/4}} + o\left(\frac{1}{q_n^{1/4}}\right), \quad \cos(\beta_n) = 1 + o\left(\frac{1}{q_n^{1/4}}\right), \quad \text{and} \quad \cot(\beta_n) = \frac{q_n^{1/4}}{2\pi} (1 + o\left(\frac{1}{q_n^{1/4}}\right)).$$

It follows that

$$\begin{aligned}
A &= \frac{2\pi(2 + \ell_4)}{q_n^{1/4}} + o\left(\frac{1}{q_n^{1/4}}\right), \quad B = 1 + o(1), \quad C = (-1 + i) \frac{\pi \ell_4}{q_n^{1/4}} + o\left(\frac{1}{q_n^{1/4}}\right), \\
F &= -\frac{2\pi^2(2\ell_4 + 1)}{q_n^{1/2}} + o\left(\frac{1}{q_n^{1/2}}\right), \quad G = \frac{q_n^{1/4}}{2\pi} - 4i + o(1), \quad H = -\frac{1}{2}(-1 + i) + o(1).
\end{aligned}$$

Return back to (4.36), we could write

$$(2 + \ell_4 + o(1))\beta_n b_1 = o(1) - \beta_n \left(-(-1 + i) \frac{2\pi^3(2\ell_4 + 1)\ell_4}{q_n^{3/4}} + o\left(\frac{1}{q_n^{3/4}}\right) \right) \sim 2\ell_4(2\ell_4 + 1)(-1 + i)\pi^3 q_n^{1/4}.$$

Hence

$$\beta_n b_1 \sim 2\ell_4 \frac{2\ell_4 + 1}{\ell_4 + 2} (-1 + i) \pi^3 q_n^{1/4}.$$

Which implies that $\|y_n^1\|$ converges to infinity as n goes to infinity and that consequently z_n is not bounded. \square

Remark 4.3. A small change in the proof leads to the conclusion that a polynomial stability, can not be better than $\frac{1}{t^2}$ (by using a frequency domain characterization of polynomial stability of a C_0 -semigroup of contraction due to Borichev and Tomilov [6]) Precisely we prove that the system is not $\frac{1}{t^\alpha}$ -polynomially stable for every α in $(0, 2)$.

4.2. A start with two fixed endpoints. (Figure 3)

$$(4.37) \quad \begin{cases} y_{tt}^j - y_{xx}^j = 0 \text{ in } (0, \ell_j) \times (0, \infty), \quad j \in \{1, 2, 3\}, \\ \sum_{j=1}^3 y_x^j(0, t) = s'(t), \quad s''(t) + s(t) = -y_t(0, t), \\ y^j(0, t) = y^l(0, t), \quad j, l \in \{1, 2, 3\}, \\ y^2(1, t) = y^3(1, t) = 0, \quad y_x^1(1, t) = -y_t(1, t), \\ s(0) = s_0, \quad s'(0) = s_1, \\ y^j(x, 0) = y_0^j(x), \quad y_t^j(x, 0) = y_1^j(x), \quad x \in (0, 1), \quad j \in \{1, 2, 3\}. \end{cases}$$

We can rewrite the system (4.37) in the Hilbert space H as

$$z'(t) = \mathcal{A}z(t),$$

where

$$H = V \times \prod_{j=1}^3 L^2(0, \ell_j) \times \mathbb{C}^2$$

with

$$V = \left\{ \Phi \in \prod_{j \in J} H^1(0, \ell_j), \quad \Phi^2(\ell_2) = \Phi^3(\ell_3) = 0, \quad \Phi^j(0) = \Phi^l(0), \quad j, l \in \{1, 2, 3\} \right\}.$$

and

$$\mathcal{A} \begin{pmatrix} y \\ v \\ p \\ q \end{pmatrix} = \begin{pmatrix} v \\ \partial_x^2 y \\ q \\ -p - v(0) \end{pmatrix}, \quad \forall \begin{pmatrix} y \\ v \\ p \\ q \end{pmatrix} \in \mathcal{D}(\mathcal{A}),$$

with

$$\begin{aligned} \mathcal{D}(\mathcal{A}) = & \left\{ (y, v, p, q) \in \left[\prod_{j=1}^3 H^2(0, \ell_j) \cap V \right] \times V \times \mathbb{C}^2; \right. \\ & \left. \sum_{j=1}^3 \frac{dy^j}{dx}(0) = q, \quad \text{and} \quad \frac{dy^1}{dx}(\ell_1) = -v^1(\ell_1) \right\}. \end{aligned}$$

The operator \mathcal{A} generates a \mathcal{C}_0 -semigroup of contraction $(S(t))_{t \geq 0}$.

If we take $\ell_1 = \ell_2 = 1$ then we have the following result.

Theorem 4.4. *the semigroup $(S(t))_{t \geq 0}$ is asymptotically stable if and only if ℓ_3 irrational and not in $\pi\mathbb{Z}$.*

Even if ℓ_3 irrational and not in $\pi\mathbb{Z}$, the semigroup $(S(t))_{t \geq 0}$ is not exponentially stable.

Proof. As in the previous case, there exists $(p_n, q_n) \in \mathbb{N}^2$ such that $\frac{p_n}{q_n}$ converge to ℓ_4 as n goes to infinity. Then q_n converge to infinity and there is a subsequence of q_n denoted q_n such that

$$|q_n \ell_3 - p_n| < \frac{1}{q_n}$$

Then we take $\beta_n = 2\pi q_n + \frac{2\pi}{q_n^{1/4}}$ and $f_n = (0, g_n, 0, 0)$ with $g_n = (0, -\sin \beta_n x, 0)$. The sequence β_n tends to infinity as n goes to infinity, $f_n \in H$ and f_n is bounded. To conclude, we prove as in the previous case that z_n defined by $(\mathcal{A} - \mathbf{i}\beta_n)z_n = f_n$ is bounded. \square

5. A chain with non equal mass points

In this section, we consider a particular network which is a chain of N edges ($N \geq 2$) and $p = N + 1$ vertices such that the $(N - 1)$ interior vertices a_j are point masses with mass m_j . But the masses m_j are not necessary equal (Figure 4).

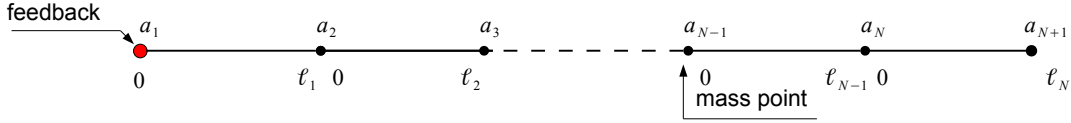


FIGURE 4. A chain of strings

Precisely, we consider the following system

$$(5.38) \quad \begin{cases} y_{tt}^j - y_{xx}^j = 0 \text{ in } (0, \ell_j) \times (0, \infty), \quad j \in \{1, \dots, N\}, \\ y_x^j(0, t) - y_x^{j-1}(\ell_{j-1}, t) = s_j'(t), \quad j \in \{2, \dots, N\}, \\ m_j s_j''(t) + s_j(t) = -y_t(0, t), \quad j \in \{2, \dots, N\}, \\ y^{j-1}(\ell_{j-1}, t) = y^j(0, t), \quad j \in \{2, \dots, N\}, \\ y_x^1(0, t) = y_t(0, t), \\ y^N(\ell_N, t) = 0, \\ s_j(0) = s_{j,0}, \quad s_j'(0) = s_{j,1}, \quad j \in \{2, \dots, N\}, \\ y^j(x, 0) = y_0^j(x), \quad y_t^j(x, 0) = y_k^1(x), \quad x \in (0, \ell_j), \quad j \in \{1, \dots, N\}. \end{cases}$$

Note that the feedback is applied at the vertex a_1 . We give a necessary and sufficient condition for the exponential stability of system (5.38). The general case of a tree with distinct masses at inner nodes is complicated for the moment.

To start we quickly redefine the associated state space H and the operator \mathcal{A} as follow:

$$H = V \times \prod_{j=1}^N L^2(0, \ell_j) \times \left(\prod_{j=1}^{N-1} \mathbb{C} \right)^2$$

with

$$V = \left\{ \Phi \in \prod_{j=1}^N H^1(0, \ell_j), \quad \Phi^N(\ell_N) = 0, \quad \Phi^{j-1}(\ell_{j-1}) = \Phi^j(0), \quad j = 2, \dots, N \right\}.$$

and

$$\mathcal{A} \begin{pmatrix} y \\ v \\ p \\ q \end{pmatrix} = \begin{pmatrix} v \\ \frac{d^2 y}{dx^2} \\ q \\ -m^{-1}(p + v_{\mathcal{M}}) \end{pmatrix}, \forall \begin{pmatrix} y \\ v \\ p \\ q \end{pmatrix} \in \mathcal{D}(\mathcal{A}),$$

with $-m^{-1}(p + v_{\mathcal{M}}) = (-\frac{1}{m_j}p_j - \frac{1}{m_j}v(\ell_j))_{j \in \{2, \dots, N-1\}}$ and

$$\mathcal{D}(\mathcal{A}) = \left\{ (y, v, p, q) \in \left[\prod_{j \in J} H^2(0, \ell_j) \cap V \right] \times V \times \prod_{k \in I_{\mathcal{M}}} \mathbb{C}^2 \text{ satisfying (5.39)} \right\}$$

where

$$(5.39) \quad \begin{cases} \frac{dy^j}{dx}(0, t) - \frac{dy^{j-1}}{dx}(\ell_{j-1}) = q_j, & j \in \{2, \dots, N\}, \\ \frac{dy^1}{dx}(0) = v^1(0). \end{cases}$$

Then the operator \mathcal{A} generates a \mathcal{C}_0 -semigroup of contractions $(S(t))_{t \in \mathbb{R}^+}$. Moreover, $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$.

For every mass point m we denote by $i_1(m), \dots, i_{k_m}(m)$ the indices of the interior nodes with masses equal to m and ordered as follow, $i_1(m) < i_2(m) < \dots < i_{k_m}(m)$.

For $r = r(m) \in \{1, \dots, k_m\}$ we define the scalars

$$\Pi_{m, r(m), s} = \sum_{i_r = j_0 < j_1 < \dots < j_s < i_{r+1}} \left(\prod_{i=0}^{s-1} c_{j_{i+1}} \sin(\beta \ell_{j_i} + \dots + \beta \ell_{j_{i+1}-1}) \right) \sin(\beta \ell_{j_s} + \dots + \beta \ell_{i_{r+1}-1})$$

and

$$\begin{aligned} \Delta_{r(m)} &: = (-1)^{i_{r+1}-i_r} \sin(\beta \ell_{i_r} + \dots + \beta \ell_{i_{r+1}-1}) \\ &\quad + \sum_{s=0}^{i_{r+1}-i_r-1} (-1)^{i_{r+1}-i_r+s} \Pi_{m, r(m), s}, \\ \text{with, } \Delta_{r(m)} &= \sin(\beta \ell_r) \text{ if } k_m = 1. \end{aligned}$$

Then we have the following result of asymptotic behavior of the system (5.38):

Lemma 5.1. *The system defined by (5.38) is exponentially stable if and only if for every mass point m and for every $r(m)$, $\Delta_{r(m)} \neq 0$.*

Proof. The first question is whether $\mathbf{i}\mathbb{R}$ belongs to $\rho(A)$. Thus, we will solve the equation

$$(5.40) \quad \mathcal{A}z = \mathbf{i}\beta z$$

of unknown $z = \begin{pmatrix} y \\ v \\ p \\ q \end{pmatrix} \in \mathcal{D}(\mathcal{A}) - \{0\}$ and $\beta \in \mathbb{R}$.

If $\beta = 0$ then $z = 0$. Thus suppose that $\beta \neq 0$. We have

$$(5.41) \quad v^j = i\beta y^j, \quad \beta^2 y^j + \frac{d^2 y^j}{dx^2} = 0, \quad j \in \{1, \dots, N\},$$

$$(5.42) \quad q^j = i\beta p^j, \quad (m_j \beta^2 - 1)p_j = v(a_j), \quad j \in \{2, \dots, N\}.$$

The function y^j , $j \in \{1, \dots, N\}$ is then of the form

$$y^j = \alpha_j \cos(\beta x) + \gamma_j \sin(\beta x).$$

If $m_j \beta^2 \neq 1$ for every j in $\{2, \dots, N\}$ then we prove by iteration, starting with $j = 1$, that $y^j = 0$ for every j in $\{2, \dots, N\}$ and consequently $z = 0$.

Now we suppose the existence of a mass point m with $m\beta^2 = 1$. Let $i_1(m) < \dots < i_{k_m}(m)$ the indices of inner nodes with masses equal to m . Then as in the first case $y^j = 0$ for every $j < i_1$. Let $r = r(m) \in \{1, \dots, k_m\}$ we have the following system

$$(5.43) \quad \begin{cases} y^r(0) = 0, \\ \text{for } j = i_r \text{ to } j = i_{r+1} - 2, y^j(\ell_j) = y^{j+1}(0), \\ \text{and } -\frac{dy^j}{dx}(\ell_j) + \frac{dy^{j+1}}{dx}(0) = q^{j+1}, \\ y^{i_{r+1}-1}(\ell_{i_{r+1}-1}) = 0 \end{cases}$$

It is obvious that the system (5.40) has trivial solution if and only if for every m and every $r(m)$ the system (5.43) has a trivial solution.

By changes of indices we can suppose that $i_r = 2$, $i_{r+1} = N + 1$. The system (5.43) can be rewritten as

$$\begin{cases} \gamma_2 = 0, \\ \text{for } j = 2 \text{ to } j = N - 1, \alpha_j \cos(\beta \ell_j) + \gamma_j \sin(\beta \ell_j) = \gamma_{j+1}, \\ \text{and } \alpha_j \sin(\beta \ell_j) - \gamma_j \cos(\beta \ell_j) + \frac{1}{\beta} \frac{1}{m_j - m} \alpha_{j+1} + \gamma_{j+1} = 0, \\ \alpha_N \cos(\beta \ell_N) + \gamma \sin(\beta \ell_N) = 0 \end{cases}$$

The matrix of such system is

$$S_N = \begin{pmatrix} 1 & 0 & 0 & & & & & & & \\ \cos x_2 & \sin x_2 & -1 & 0 & & & & & & \\ \sin x_2 & -\cos x_2 & c_3 & 1 & 0 & & & & & \\ 0 & 0 & \cos x_3 & \sin x_3 & -1 & 0 & & & & \\ 0 & 0 & \sin x_3 & -\cos x_3 & c_4 & 1 & 0 & & & \\ & & & & \dots & \dots & \dots & \dots & & \\ & & & & & & \dots & \dots & & \\ & & & & & & & & 0 & \\ & & & & & & & \cos x_{N-1} & \sin x_{N-1} & -1 & 0 \\ & & & & & & & \sin x_{N-1} & -\cos x_{N-1} & c_N & 1 \\ & & & & & & & 0 & 0 & \cos x_N & \sin x_N \end{pmatrix} \quad (0)$$

where $x_j = \beta \ell_j$ and $c_j = \frac{1}{\beta(m_j - m)}$. We want to calculate the determinant Δ_N of S_N . For this, Let M_N the determinant of the matrix obtain from S_N by replacing $\cos x_N$ and $\sin x_N$ in the last line by $\sin x_N$ and $-\cos x_N$ respectively.

One can verifies easily that

$$\begin{aligned}
\Delta_2 &= \sin x_2, \quad \Delta_3 = -\sin(x_2 + x_3) + c_3 \sin x_2 \sin x_3, \\
\Delta_4 &= \sin(x_2 + x_3 + x_4) - c_3 \sin x_2 \sin(x_3 + x_4) - c_4 \sin(x_2 + x_3) \sin x_4 \\
&\quad + c_3 c_4 \sin x_2 \sin x_3 \sin x_4, \\
M_2 &= -\cos x_2, \quad M_3 = \cos(x_2 + x_3) - c_3 \sin x_2 \cos x_3, \\
M_4 &= -\cos(x_2 + x_3 + x_4) + c_3 \sin x_2 \cos(x_3 + x_4) + c_4 \sin(x_2 + x_3) \cos x_4 \\
&\quad - c_3 c_4 \sin x_2 \sin x_3 \cos x_4.
\end{aligned}$$

We will prove by induction that for every integer $N \geq 2$,

$$\Delta_N = (-1)^N \sin(x_2 + \dots + x_N) + (-1)^{N-1} \sum_{j=2}^{N-1} c_{j+1} \sin(x_2 + \dots + x_{j-1}) \sin(x_j + \dots + x_N)$$

(5.44)

$$+ \sum_{s=2}^{N-2} (-1)^{N-1-s} \sum_{2=j_0 < j_1 < \dots < j_s \leq N} \left(\prod_{i=0}^{s-1} c_{j_{i+1}} \sin(x_{j_i} + \dots + x_{j_{i+1}-1}) \right) \sin(\beta \ell_{j_s} + \dots + \beta \ell_N)$$

and

$$M_N = (-1)^{N+1} \cos(x_2 + \dots + x_N) + (-1)^N \sum_{j=2}^{N-1} c_{j+1} \sin(x_2 + \dots + x_{j-1}) \cos(x_j + \dots + x_N)$$

(5.45)

$$+ \sum_{s=2}^{N-2} (-1)^{N-s} \sum_{2=j_0 < j_1 < \dots < j_s \leq N} \left(\prod_{i=0}^{s-1} c_{j_{i+1}} \sin(x_{j_i} + \dots + x_{j_{i+1}-1}) \right) \cos(x_{j_s} + \dots + x_N).$$

Such rules are true for $N = 2$. and $N = 3$ let $N \in \mathbb{N}$ with $N \geq 3$ and suppose that (5.44) and (5.45) are true. Some calculations leads to

$$\begin{aligned}
\Delta_N &= (-\cos x_N + c_N \sin x_N) \Delta_{N-1} + (\sin x_N) M_{N-1}, \\
M_N &= (-\sin x_N - c_N \cos x_N) \Delta_{N-1} - (\cos x_N) M_{N-1}
\end{aligned}$$

We can now verify the rule (5.44) to order N :

$$\begin{aligned}
&\Delta_N \\
&= \left((-1)^N \sin(x_2 + \dots + x_N) + (-1)^{N-1} \sum_{j=2}^{N-1} c_{j+1} \sin(x_2 + \dots + x_{j-1}) \sin(x_j + \dots + x_N) \right. \\
&\quad + \sum_{s=2}^{N-3} (-1)^{N-1-s} \sum_{2=j_0 \leq j_1 < \dots < j_s \leq N-1} \left(\prod_{i=0}^{s-1} c_{j_{i+1}} \sin(x_{j_i} + \dots + x_{j_{i+1}-1}) \right) \sin(x_{j_s} + \dots + x_N) \Big) \\
&\quad + \sum_{s=2}^{N-3} (-1)^{N-2-s} \sum_{2=j_0 \leq j_1 < \dots < j_s \leq N-1} c_N \left(\prod_{i=0}^{s-1} c_{j_{i+1}} \sin(x_{j_i} + \dots + x_{j_{i+1}-1}) \right) \sin(x_{j_s} + \dots + x_{N-1}) \sin x_N \Big) \\
&= \left((-1)^N \sin(x_2 + \dots + x_N) + (-1)^{N-1} \sum_{j=2}^{N-1} c_{j+1} \sin(x_2 + \dots + x_{j-1}) \sin(x_j + \dots + x_N) \right. \\
&\quad + \sum_{s=2}^{N-2} (-1)^{N-1-s} \sum_{2=j_0 \leq j_1 < \dots < j_s \leq N} \left(\prod_{i=0}^{s-1} c_{j_{i+1}} \sin(x_{j_i} + \dots + x_{j_{i+1}-1}) \right) \sin(x_{j_s} + \dots + x_N) \Big).
\end{aligned}$$

A similar calculus, using (5.45) proves that (5.45) is verified in order N .

We can now state the following results:

The associated semigroup $S(t)$ is asymptotically stable if and only if $\Delta_{r(m)}$ is different from zero for every mass point m and every $r(m)$. To conclude that $(S(t))_{t \geq 0}$ is exponentially stable, it suffices to prove that (3.20) is satisfied by $(S(t))_{t \geq 0}$, exactly as in the proof of Theorem 3.2 \square

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